Compact operators without extended eigenvalues

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Abstract

A complex number λ is called an extended eigenvalue of a bounded linear operator T on a Banach space $\mathcal{B}$ if there exists a non-zero bounded linear operator X acting on $\mathcal{B}$ such that $XT = \lambda TX$. We show that there are compact quasinilpotent operators on a separable Hilbert space, for which the set of extended eigenvalues is the one-point set $\{1\}$.

1 Introduction

All vector spaces in this article are over the field $\mathbb{C}$ of complex numbers. For a Banach space $\mathcal{B}$, $L(\mathcal{B})$ stands for the algebra of bounded linear operators on $\mathcal{B}$. A complex number λ is called an *extended eigenvalue* of $T \in L(\mathcal{B})$ if there exists non-zero $X \in L(\mathcal{B})$ such that $XT = \lambda TX$. We denote the set of extended eigenvalues of T by the symbol $\Sigma(T)$. Extended eigenvalues and their corresponding extended eigenoperators X as well as their applications to classification of invariant subspaces of linear operators were studied in [2, 3, 6]. Obviously $1 \in \Sigma(T)$ for any operator T. Indeed, one can take X being the identity operator.

Let V be the Volterra operator acting on $L_2[0,1]$:

$$V : L_2[0,1] \to L_2[0,1], \quad Vf(x) = \int_0^1 f(t)\,dt.$$

In [2] the set of extended eigenvalues of V has been computed.

Theorem **BLP.** *The set $\Sigma(V)$ coincides with the set $\mathbb{R}^+$ of positive real numbers.*

In [6] it is also shown that $\Sigma(\gamma I + V) = \{1\}$ for each non-zero $\gamma \in \mathbb{C}$. Note that $\gamma I + V$ is neither compact nor quasinilpotent. The following questions were raised in [2]. In what follows $\mathcal{H}$ stands for a separable infinite dimensional Hilbert space.

Question **1.** *Does there exist a compact operator $T \in L(\mathcal{H})$ for which $\Sigma(T) = \{1\}$?*

Question **2.** *Does there exist a quasinilpotent operator $T \in L(\mathcal{H})$ for which $\Sigma(T) = \{1\}$?*

We answer both questions affirmatively:

Theorem **1.1.** *There exists a compact quasinilpotent operator $T \in L(\mathcal{H})$ for which $\Sigma(T) = \{1\}$.*

It is worth noting that $\Sigma(T) = \mathbb{C}$ if T is a nilpotent operator acting on a Banach space $\mathcal{B}$. Indeed, if $T = 0$, then $XT = \lambda TX = 0$ for each $X \in L(\mathcal{B})$ and each $\lambda \in \mathbb{C}$. If $T \neq 0$, there

exists a positive integer n such that $T^n \neq 0$ and $T^{n+1} = 0$. Then $XT = \lambda TX = 0$ is satisfied with $X = T^n$ for any $\lambda \in \mathbb{C}$.

The proof of Theorem 1.1 is non-constructive. The existence of a required operator T is proved by using the Baire category argument. The proof is based on a theorem of Apostol, characterizing closures of similarity orbits of compact quasinilpotent operators and precise knowledge of the sets of extended eigenvalues for two specific operators, one of which is the Volterra operator and the other one is a bilateral weighted shift.

We compute extended eigenvalues of bilateral weighted shifts and prove other auxiliary results in Section 2. Theorem 1.1 is proved in Section 3. In Section 4 of concluding remarks we discuss previous results and raise few problems.

2 Auxiliary results

From now on $\mathbb{Z}$ stands for the set of integers and $\mathbb{Z}_+$ is the set of non-negative integers.

Proposition **2.1.** *Let $1 \leqslant p < \infty$, $w = \{w_n\}_{n\in\mathbb{Z}}$ be a bounded sequence of non-zero complex numbers and T be the bilateral weighted shift with the weight sequence w acting on the space $\ell_p(\mathbb{Z})$, that is $Te_n = w_n e_{n-1}$ for any $n \in \mathbb{Z}$, where $\{e_n\}_{n\in\mathbb{Z}_+}$ is the canonical basis of $\ell_p(\mathbb{Z})$. Let also*

$$\beta(k,n) = \prod_{j=k}^{n} w_j \ \text{ for } \ k,n \in \mathbb{Z},\ k \leqslant n \ \text{ and } \ \beta(n+1,n) = 1 \ \text{ for } n \in \mathbb{Z}. \tag{1}$$

Then $0 \notin \Sigma(T)$ and a non-zero complex number λ belongs to $\Sigma(T)$ if and only if there exists $k \in \mathbb{Z}_+$ such that the sequence $\{\lambda^{-n}\beta(n-k+1,n)\}_{n\in\mathbb{Z}}$ is bounded.

Proof. Clearly T is injective and has dense range. Therefore zero is not an extended eigenvalue of T. First, assume that there exists $k \in \mathbb{Z}_+$ for which the sequence $\{\lambda^{-n}\beta(n-k+1,n)\}_{n\in\mathbb{Z}}$ is bounded. Then there exists a unique bounded linear operator X on $\ell_p(\mathbb{Z})$ such that $Xe_n = \lambda^{-n}\beta(n-k+1,n)e_{n-k}$ for any $n \in \mathbb{Z}$. Clearly $X \neq 0$. It is straightforward to verify that $XT = \lambda TX$. Hence $\lambda \in \Sigma(T)$.

Assume now that the sequence $\{\lambda^{-n}\beta(n-k+1,n)\}_{n\in\mathbb{Z}}$ is unbounded for any $k \in \mathbb{Z}_+$. Let $X \in L(\ell_p(Z))$ be such that $XT = \lambda TX$. We have to prove that $X = 0$. Let $\{x_{n,j}\}_{n,j\in\mathbb{Z}}$ be the matrix of X, that is a complex matrix such that

$$Xe_n = \sum_{j\in\mathbb{Z}} x_{n,j}e_j \ \text{ for any } n \in \mathbb{Z}_+.$$

From the equation $XT = \lambda TX$ it immediately follows that $w_n x_{n-1,j} = \lambda w_{j+1} x_{n,j+1}$ for any $j,n \in \mathbb{Z}$. Therefore

$$x_{n,n+k} = \begin{cases} \lambda^{-n}\frac{\beta(1,k)}{\beta(n+1,n+k)}x_{0,k} & \text{if } k > 0; \\ \lambda^{-n}\frac{\beta(k+n+1,n)}{\beta(k+1,0)}x_{0,k} & \text{if } k \leqslant 0 \end{cases} \qquad \text{for any } n,k \in \mathbb{Z}. \tag{2}$$

Boundedness of X implies boundedness of the set $\{x_{n,j} : n,j \in \mathbb{Z}\}$. If there exists $k \leqslant 0$ for which $x_{0,k} \neq 0$, then according to (2), boundedness of $\{x_{n,n+k}\}_{n\in\mathbb{Z}}$ implies boundedness of $\{\lambda^{-n}\beta(k+n+1,n)\}_{n\in\mathbb{Z}}$, which is unbounded by assumption. If there exists $k > 0$ for which $x_{0,k} \neq 0$, then according to (2) boundedness of $\{x_{n,n+k}\}_{n\in\mathbb{Z}}$ implies boundedness of $\{\lambda^{-n}/\beta(n+1,n+k)\}_{n\in\mathbb{Z}}$. Since $\{w_n\}_{n\in\mathbb{Z}}$ is bounded, the sequence $\{\beta(n+1,n+k)\}_{n\in\mathbb{Z}}$ is also bounded, and it follows that $\{\lambda^{-n} = \lambda^{-n}\beta(n+1,n)\}_{n\in\mathbb{Z}}$ is bounded as a product of two bounded sequences, which also

contradicts the assumption. Thus, $x_{0,k}=0$ for each $k\in\mathbb{Z}$. Formula (2) now implies that $x_{n,j}=0$ for any $n,j\in\mathbb{Z}$ and therefore $X=0$. $\Box$

REMARK **1**. Let for $k\in\mathbb{Z}_+$,

$$c_k=\varlimsup_{n\to+\infty}\bigl|\beta(n+1,n+k)\bigr|^{1/n}\ \text{ and }\ d_k=\varliminf_{n\to+\infty}\bigl|\beta(1-n,k-n)\bigr|^{-1/n}.$$

Then $c_{k+1}\leqslant c_k\leqslant 1\leqslant d_k\leqslant d_{k+1}$ for each $k\in\mathbb{Z}_+$. Denote $c=\lim\limits_{k\to\infty}c_k$ and $d=\lim\limits_{k\to\infty}d_k$. Then $0\leqslant c\leqslant 1\leqslant d\leqslant\infty$. From Proposition 2.1 it follows that $\Sigma(T)$ is an annulus of one of the following shapes $\{z\in\mathbb{C}:c\leqslant|z|\leqslant d\}$, $\{z\in\mathbb{C}:c<|z|\leqslant d\}$, $\{z\in\mathbb{C}:c\leqslant|z|<d\}$ or $\{z\in\mathbb{C}:c<|z|<d\}$. Moreover, $\Sigma(T)$ always contains the unit circle.

COROLLARY **2.2.** *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Then there exists an injective quasinilpotent compact operator T on $\mathcal{H}$ with dense range such that $\Sigma(T)$ coincides with the unit circle $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$.*

Proof. Since all separable infinite dimensional Hilbert spaces are isomorphic, we can assume that $\mathcal{H}=\ell_2(\mathbb{Z})$. Let $T:\mathcal{H}\to\mathcal{H}$ be the bilateral weighted shift with the weight sequence $w_n=(1+|n|)^{-1}$. Then we have $\beta(n-k+1,n)\sim(1+|n|)^{-k}$ as $n\to\infty$ for each $k\in\mathbb{Z}_+$, where the numbers $\beta(a,b)$ are defined by (1). Hence for any non-zero complex number λ and any $k\in\mathbb{Z}_+$, the sequence $\{\lambda^{-n}\beta(n-k+1,n)\}_{n\in\mathbb{Z}}$ is bounded if and only if $|\lambda|=1$. According to Proposition 2.1, $\Sigma(T)=\mathbb{T}$. Clearly T is compact, injective and has dense range. Since for any $k\in\mathbb{Z}_+$ and any $n\in\mathbb{Z}$, $T^ke_n=\beta(n-k+1,n)e_{n-k}$, we have

$$\|T^k\|=\max_{n\in\mathbb{Z}}\beta(n-k+1,n). \tag{3}$$

Now, if $\alpha_n=\beta(n-k+1,n)$, then according to (1),

$$\frac{\alpha_n}{\alpha_{n-1}}=\frac{w_n}{w_{n-k}}=\frac{1+|n-k|}{1+|n|}.$$

Therefore $\frac{\alpha_n}{\alpha_{n-1}}\leqslant 1$ if $n\geqslant k/2$ and $\frac{\alpha_n}{\alpha_{n-1}}\geqslant 1$ if $n\leqslant k/2$. Thus, α_n is maximal for $n=[k/2]$, where $[k/2]$ is the integer part of $k/2$. Hence

$$\max_{n\in\mathbb{Z}}\beta(n-k+1,n)=\max_{n\in\mathbb{Z}}\alpha_n=\alpha_{[k/2]}=\begin{cases}(m!)^{-2},\ \text{where } m=(k+1)/2 & \text{if } k \text{ is odd;}\\ (m!(m+1)!)^{-1},\ \text{where } m=k/2 & \text{if } k \text{ is even.}\end{cases}$$

The above display together with (3) implies $\lim\limits_{k\to\infty}\|T^k\|^{1/k}=0$. Thus, from the spectral radius formula [7] it follows that T is quasinilpotent. $\Box$

The following lemma is a direct consequence of Theorem BLP.

LEMMA **2.3.** *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Then there exists an injective quasinilpotent compact operator T on $\mathcal{H}$ with dense range such that $\Sigma(T)$ coincides with the open half-line $\mathbb{R}^+=\{x\in\mathbb{R}:x>0\}$.*

For a Banach space $\mathcal{B}$, symbol $\Omega(\mathcal{B})$ stands for the set of compact quasinilpotent operators $T\in L(\mathcal{B})$ endowed with the operator norm metric $d(T,S)=\|T-S\|$. It is easy to show that $(\Omega(\mathcal{B}),d)$ is a complete metric space, or equivalently, that $\Omega(\mathcal{B})$ is closed in the Banach space $K(\mathcal{B})$ of compact operators on $\mathcal{B}$. Indeed, if $T\in K(\mathcal{B})$ non-quasinilpotent, then T has a non-zero normal eigenvalue. Hence all operators sufficiently close to T with respect to the metric d also have a non-zero normal eigenvalue and therefore can not be quasinilpotent. It is worth noting that we actually need completeness of $\Omega(\mathcal{B})$ only in the case when $\mathcal{B}=\mathcal{H}$ is a Hilbert space. In this case it can be found, for instance, in [5].

For a bounded linear operator T acting on a Banach space $\mathcal{B}$ we denote

$$\mathrm{Sim}\,(T)=\{GTG^{-1}:G\in L(\mathcal{B})\ \text{ is invertible.}\}\tag{4}$$

The set $\mathrm{Sim}\,(T)$ is usually called the *similarity orbit* of T. The key tool for the proof of Theorem 1.1 is the following theorem of Apostol, which is the main result in [1]. It worth noting that Apostol's theorem answers a question raised by Herrero in [4], where a weaker statement was proved.

THEOREM **A.** *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and $T\in\Omega(\mathcal{H})$ be non-nilpotent. Then $\mathrm{Sim}\,(T)$ is dense in $\Omega(\mathcal{H})$.*

Recall that a subset A of a topological space Y is called an F_σ-set if A is a union of countably many closed subsets of Y. The complement of an F_σ-set is called a G_δ-set. Equivalently a G_δ-set in Y is a countable intersection of open subsets of Y.

LEMMA **2.4.** *Let $\mathcal{B}$ be a separable reflexive Banach space and $A\subset\mathbb{C}$ be an F_σ-set. Then*

$$M(A)=\{T\in L(\mathcal{B}):\Sigma(T)\cap A\neq\varnothing\}$$

is an F_σ-set in $L(\mathcal{B})$ endowed with the topology defined by the operator norm metric $d(T,S)=\|T-S\|$.

Proof. Obviously any F_σ-set in $\mathbb{C}$ is a union of countably many compact sets. Pick compact sets $K_n\subset\mathbb{C}$, $n\in\mathbb{Z}_+$ such that $A=\bigcup\limits_{k=0}^{\infty}K_n$. Since $\mathcal{B}$ is separable, there exists a sequence $\{x_m\}_{m\in\mathbb{Z}_+}$ of elements of $\mathcal{B}$ such that the set $\{x_m:m\in\mathbb{Z}_+\}$ is dense in $\mathcal{B}$. Reflexivity of $\mathcal{B}$ implies that $\mathcal{B}^*$ is also separable and we can choose a sequence $\{y_k\}_{k\in\mathbb{Z}_+}$ of elements of $\mathcal{B}^*$ such that $\{y_k:k\in\mathbb{Z}_+\}$ is dense in $\mathcal{B}^*$. For $n,m,k,j\in\mathbb{Z}_+$ we denote

$$N(n,m,k,j)=\left\{T\in L(B):\ \begin{array}{l}\text{there exist } X\in L(\mathcal{B}) \text{ and } \lambda\in K_n \text{ such that}\\ XT=\lambda TX,\ \|X\|\leqslant j \text{ and } \langle Xx_m,y_k\rangle=1\end{array}\right\}.$$

First, we shall show that

$$M(A)=\bigcup_{n,m,k,j\in\mathbb{Z}_+}N(n,m,k,j).\tag{5}$$

The inclusion $N(n,m,k,j)\subseteq M(A)$ follows immediately from the definitions of $N(n,m,k,j)$ and $M(A)$. Let $T\in M(A)$. Then there exists $\lambda\in A$ and non-zero $Y\in L(\mathcal{B})$ such that $YT=\lambda TY$. Since $A=\bigcup\limits_{k=0}^{\infty}K_n$, there exists $n\in\mathbb{Z}_+$ such that $\lambda\in K_n$. Since $Y\neq 0$ and $\{x_m:m\in\mathbb{Z}_+\}$ is dense in $\mathcal{B}$, there exists $m\in\mathbb{Z}_+$ for which $Yx_m\neq 0$. Since $\{y_k:k\in\mathbb{Z}_+\}$ is dense in $\mathcal{B}^*$, there exists $k\in\mathbb{Z}_+$ such that $\langle Yx_m,y_k\rangle\neq 0$. Let $X=(\langle Yx_m,y_k\rangle)^{-1}Y$. Then $XT=\lambda TX$ and $\langle Xx_m,y_k\rangle=1$. Finally taking $j\in\mathbb{Z}_+$ such that $j\geqslant\|X\|$, we ensure that $T\in N(n,m,k,j)$, which proves (5).

In view of (5), it suffices to verify that the sets $N(n,m,k,j)$ are closed with respect to the operator norm topology. Let $n,m,k,j\in\mathbb{Z}_+$, $T\in L(\mathcal{B})$ and $\{T_l\}_{l\in\mathbb{Z}_+}$ be a sequence of elements of $N(n,m,k,j)$ such that $\|T_l-T\|\to 0$ as $l\to\infty$. We have to verify that $T\in N(n,m,k,j)$. Since $T_l\in N(n,m,k,j)$, there exist $\lambda_l\in K_n$ and $X_l\in L(\mathcal{B})$ such that $T_lX_l=\lambda_lX_lT_l$, $\|X_l\|\leqslant j$ and $\langle X_lx_m,y_k\rangle=1$ for each $l\in\mathbb{Z}_+$. Since $\mathcal{B}$ is separable and reflexive for any $c>0$ and sequence $\{Z_l\}_{l\in\mathbb{Z}_+}$ of elements of $L(\mathcal{B})$ such that $\|Z_l\|\leqslant c$ for each $l\in\mathbb{Z}_+$, there exists $Z\in L(\mathcal{B})$ with $\|Z\|\leqslant c$ and a strictly increasing sequence $\{l_q\}_{q\in\mathbb{Z}_+}$ of positive integers such that Z_{l_q} converges to Z as $q\to\infty$ with respect to the weak operator topology [7], that is $Z_{l_q}x$ converges weakly to Zx for each $x\in\mathcal{B}$. Using this fact along with compactness of K_n, we can, passing to a subsequence, if necessary, assume that $\lambda_l\to\lambda\in K_n$ and $X_l\to X\in L(\mathcal{B})$ as $l\to\infty$ with respect to the

weak operator topology, where $\|X\| \leqslant j$. Since $\langle X_l x_m, y_k \rangle = 1$ for each $l \in \mathbb{Z}_+$, we see that $\langle X x_m, y_k \rangle = 1$. Let $x \in \mathcal{B}$. Then

$$(T_l X_l - \lambda_l X_l T_l)x = (\lambda - \lambda_l)X_l T_l x + (T - T_l)X_l x + \lambda X_l (T - T_l)x + TX_l x - \lambda X_l T x. \qquad (6)$$

Since the sequence T_l is norm-convergent, there exists $c > 0$ such that $\|T_l\| \leqslant c$ for each $l \in \mathbb{Z}_+$. Then

$$\begin{aligned} \|(\lambda - \lambda_l)X_l T_l x\| &\leqslant jc\|x\||\lambda - \lambda_l| \to 0 \qquad && \text{as } l \to \infty, \\ \|(T - T_l)X_l x\| &\leqslant j\|x\|\|T - T_l\| \to 0 && \text{as } l \to \infty, \\ \|\lambda X_l (T - T_l)x\| &\leqslant j|\lambda|\|x\|\|T - T_l\| \to 0 && \text{as } l \to \infty. \end{aligned}$$

Since $X_l x$ converge weakly to Xx and any bounded linear operator on a Banach space is weak-to-weak continuous, we see that $TX_l x$ converge weakly to TXx. Finally $X_l T x$ converge weakly to XTx. Thus, from (6) and the last display it follows that $(T_l X_l - \lambda_l X_l T_l)x$ converge weakly to $(TX - \lambda XT)x$. On the other hand $(T_l X_l - \lambda_l X_l T_l)x = 0$ for any $l \in \mathbb{Z}_+$ and therefore $(TX - \lambda XT)x = 0$ for each $x \in \mathcal{B}$. Thus, $TX - \lambda XT = 0$, which proves the inclusion $T \in N(n, m, k, j)$. $\Box$

3 Proof of Theorem 1.1

Let $A_1 = \mathbb{C} \setminus \mathbb{R}^+$ and $A_2 = \mathbb{C} \setminus \mathbb{T}$. Clearly A_1 and A_2 are F_σ-sets in $\mathbb{C}$. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and

$$M_j = \{T \in \Omega(\mathcal{H}) : \Sigma(T) \cap A_j \neq \varnothing\}, \quad j = 1, 2.$$

According to Lemma 2.4 M_j are F_σ-sets in $\Omega(\mathcal{H})$. By Lemma 2.3 and Corollary 2.2, there exist non-nilpotent operators $T_1, T_2 \in \Omega(\mathcal{H})$ such that $\Sigma(T_1) = \mathbb{R}_+$ and $\Sigma(T_2) = \mathbb{T}$. Since the set $\Sigma(T)$ is a similarity invariant, we have that $\Sigma(S) = \mathbb{R}_+$ for each $S \in \mathrm{Sim}\,(T_1)$ and $\Sigma(S) = \mathbb{T}$ for each $S \in \mathrm{Sim}\,(T_2)$. Hence

$$\mathrm{Sim}\,(T_1) \cap M_1 = \mathrm{Sim}\,(T_2) \cap M_2 = \varnothing.$$

By Theorem A, both $\mathrm{Sim}\,(T_1)$ and $\mathrm{Sim}\,(T_2)$ are dense in $\Omega(\mathcal{H})$. According to the last display, $\Omega(\mathcal{H}) \setminus M_1$ and $\Omega(\mathcal{H}) \setminus M_2$ are both dense G_δ sets in $\Omega(\mathcal{H})$. Since $\Omega(\mathcal{H})$ is a complete metric space, the Baire theorem implies that

$$(\Omega(\mathcal{H}) \setminus M_1) \cap (\Omega(\mathcal{H}) \setminus M_2) = \Omega(\mathcal{H}) \setminus (M_1 \cup M_2)$$

is a dense G_δ-set in $\Omega(\mathcal{H})$ and therefore $M_1 \cup M_2$ is a Baire first category set, that is a countable union of nowhere dense sets. From the definitions of M_1 and M_2 and the equality $\mathbb{R}^+ \cap \mathbb{T} = \{1\}$ it immediately follows that

$$M_1 \cup M_2 = \{T \in \Omega(\mathcal{H}) : \Sigma(T) \neq \{1\}\}.$$

Thus, we have proven the following theorem.

Theorem 3.1. *Let $\mathcal{H}$ be a separable Hilbert space. Then $\{T \in \Omega(\mathcal{H}) : \Sigma(T) \neq \{1\}\}$ is a Baire first category set in the complete metric space $\Omega(\mathcal{H})$.*

Theorem 1.1 is an immediate consequence of Theorem 3.1 and the Baire theorem.

4 Concluding remarks

Theorem BLP shows that the set $\Sigma(T)$ for a bounded linear operator on a separable Hilbert space can be non-closed. The following proposition shows that the sets $\Sigma(T)$ can not be too bad. The proof goes similarly to the proof of Lemma 2.4. It is included for sake of completeness.

PROPOSITION **4.1.** *Let $\mathcal{B}$ be a separable reflexive Banach space and $T \in L(\mathcal{B})$. Then $\Sigma(T)$ is an F_σ-set.*

Proof. As in the proof of Lemma 2.4, we can choose sequences $\{x_m\}_{m\in\mathbb{Z}_+}$ of elements of $\mathcal{B}$ and $\{y_k\}_{k\in\mathbb{Z}_+}$ of elements of $\mathcal{B}^*$ such that $\{x_m : m \in \mathbb{Z}_+\}$ is dense in $\mathcal{B}$ and $\{y_k : k \in \mathbb{Z}_+\}$ is dense in $\mathcal{B}^*$. For $m, k, j \in \mathbb{Z}_+$ we denote

$$A(m,k,j) = \left\{ \lambda \in \mathbb{C} : \begin{array}{l} \text{there exist } X \in L(\mathcal{B}) \text{ such that } XT = \lambda TX, \\ \|X\| \leqslant j \text{ and } \langle Xx_m, y_k\rangle = 1 \end{array} \right\}.$$

First, we shall show that

$$\Sigma(T) = \bigcup_{m,k,j\in\mathbb{Z}_+} A(m,k,j). \tag{7}$$

The inclusion $A(m,k,j) \subseteq \Sigma(T)$ is obviously valid for any $m, k, j \in \mathbb{Z}_+$. Let $\lambda \in \Sigma(T)$. Then there exists non-zero $Y \in L(\mathcal{B})$ such that $YT = \lambda TY$. Since $Y \neq 0$ and $\{x_m : m \in \mathbb{Z}_+\}$ is dense in $\mathcal{B}$, there exists $m \in \mathbb{Z}_+$ for which $Yx_m \neq 0$. Since $\{y_k : k \in \mathbb{Z}_+\}$ is dense in $\mathcal{B}^*$, there exists $k \in \mathbb{Z}_+$ such that $\langle Yx_m, y_k\rangle \neq 0$. Let $X = (\langle Yx_m, y_k\rangle)^{-1}Y$. Then $XT = \lambda TX$ and $\langle Xx_m, y_k\rangle = 1$. Finally taking $j \in \mathbb{Z}_+$ such that $j \geqslant \|X\|$, we ensure that $\lambda \in A(m,k,j)$, which proves (7).

In view of (7), it suffices to verify that the sets $A(m,k,j)$ are closed in $\mathbb{C}$. Let $m, k, j \in \mathbb{Z}_+$, $\lambda \in \mathbb{C}$ and $\{\lambda_l\}_{l\in\mathbb{Z}_+}$ be a sequence of elements of $A(m,k,j)$, converging to λ. Since $\lambda_l \in A(m,k,j)$, there exist $X_l \in L(\mathcal{B})$ such that $TX_l = \lambda_l X_l T$, $\|X_l\| \leqslant j$ and $\langle X_l x_m, y_k\rangle = 1$ for each $l \in \mathbb{Z}_+$. Since $\mathcal{B}$ is separable and reflexive, we, passing to a subsequence, if necessary, assume that $X_l \to X \in L(\mathcal{B})$ as $l \to \infty$ with respect to the weak operator topology, where $\|X\| \leqslant j$. Since $\langle X_l x_m, y_k\rangle = 1$ for each $l \in \mathbb{Z}_+$, we see that $\langle Xx_m, y_k\rangle = 1$. Let $x \in \mathcal{B}$. Then

$$(TX_l - \lambda_l X_l T)x = (\lambda - \lambda_l)X_l Tx + TX_l x - \lambda X_l Tx. \tag{8}$$

Clearly

$$\|(\lambda - \lambda_l)X_l Tx\| \leqslant j|\lambda - \lambda_l|\|T\|\|x\| \to 0 \quad \text{as } l \to \infty.$$

Since $X_l x$ converge weakly to Xx and any bounded linear operator on a Banach space is weak-to-weak continuous, we see that $TX_l x$ converge weakly to TXx. Finally $X_l Tx$ converge weakly to XTx. Thus, from (8) and the last display it follows that $(TX_l - \lambda_l X_l T)x$ converge weakly to $(TX - \lambda XT)x$. On the other hand $(TX_l - \lambda_l X_l T)x = 0$ for any $l \in \mathbb{Z}_+$ and therefore $(TX - \lambda XT)x = 0$ for each $x \in \mathcal{B}$. Hence $TX - \lambda XT = 0$, which proves the inclusion $\lambda \in A(m,k,j)$. □

Proposition 4.1 is a first step towards the solution of the following problem.

QUESTION **4.2.** *Which subsets of $\mathbb{C}$ have the form $\Sigma(T)$ for $T \in L(\mathcal{H})$? Which subsets of $\mathbb{C}$ have the form $\Sigma(T)$ for compact $T \in L(\mathcal{H})$? Which subsets of $\mathbb{C}$ have the form $\Sigma(T)$ for compact quasinilpotent subsets of $\mathbb{C}$? What about operators on Banach spaces?*

The key point of the above proof of Theorem 1.1 is application of Theorem A, which does not work in the Banach space setting. This leads naturally to the following question.

QUESTION **4.3.** *For which Banach spaces $\mathcal{B}$ there exists a compact quasinilpotent operator $T \in L(\mathcal{B})$ such that $\Sigma(T) = \{1\}$?*

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